TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 323, Number 1, January 1991

SOBOLEV INTERPOLATION INEQUALITIES WITH WEIGHTS

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ABSTRACT. We study weighted local Sobolev interpolation inequalities of the form

$$\frac{1}{w_2(B)}\int_B |u(x)|^{ph} w_2(x)\,dx \le c\left(\frac{1}{v(B)}\int_B |u(x)|^p v(x)\,dx\right)^{h-1} \times\left(\frac{|B|^{p/n}}{w_1(B)}\int_B |\nabla u(x)|^p w_1(x)\,dx + \frac{1}{v(B)}\int_B |u(x)|^p v(x)\,dx\right),$$

where $1 < p < \infty$, $h > 1$, B is a ball in $\mathbf{R}^n$, and v, w_1, and w_2 are weight functions. The case $p = 2$ is of special importance in deriving regularity results for solutions of degenerate parabolic equations. We also study the analogous inequality without the second summand on the right in the case u has compact support in B, and we derive global Landau inequalities $\|\nabla u\|_{L^q_w} \le c\|u\|_{L^p_v}^{1-a}\|\nabla^2 u\|_{L^p_v}^a$, $0 < a < 1$, $1 < p \le q < \infty$, when u has compact support.

1. INTRODUCTION

In this paper we consider interpolation inequalities of the form

$$\frac{1}{w_2(B)}\int_B |u(x)|^{2h} w_2(x)\,dx \le c\left(\frac{1}{v(B)}\int_B u(x)^2 v(x)\,dx\right)^{h-1} \times\left(\frac{|B|^{2/n}}{w_1(B)}\int_B |\nabla u(x)|^2 w_1(x)\,dx + \frac{1}{v(B)}\int_B u(x)^2 v(x)\,dx\right), \tag{1.1}$$

for $u \in \mathrm{Lip}(\overline{B})$. Here $h > 1$, B is a ball in $\mathbf{R}^n$, $|B|$ denotes the Lebesgue measure of B, $v(B) = \int_B v(x)\,dx$, v, w_1, and w_2 are nonnegative functions, and $c > 0$ is a constant independent of B and u. Also, $\mathrm{Lip}(\overline{B})$ denotes the class of functions which are Lipschitz continuous in $\overline{B}$, and $\mathrm{Lip}_0(B)$ denotes the subclass of functions with compact support in B. We also derive inequalities analogous to (1.1) with the second summand on the right-hand side omitted if u has compact support in B.

Such inequalities are the main tool in proving Harnack's inequality for solutions of certain degenerate parabolic equations by using the iteration method of Moser [12]. For this method it is important to have $h > 1$ on the left-hand side

Received by the editors January 4, 1989.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 46E35.
The second author was supported in part by NSF Grant DMS-87-03546.

of (1.1). Using the results in this paper we show in [8] Harnack's inequality for solutions of equations of the form

$$v(x)\,u_t = \sum_{i,j=1}^{n} D_{x_i}(a_{ij}(x,t)\,D_{x_j}u),$$

where

$$w_1(x)\,|\xi|^2 \le \sum_{i,j=1}^{n} a_{ij}(x,t)\,\xi_i\xi_j \le w_2(x)\,|\xi|^2,$$

$x \in \Omega$, $t \in \mathbf{R}$, with Ω a domain in $\mathbf{R}^n$. We mention that such equations arise by making a quasiconformal change of variable $y = \varphi(x)$ in the heat equation $\frac{\partial}{\partial t} = \Delta_y$; in this case, $v = |\det(\varphi')|$ and w_1 and w_2 are constant multiplies of $v^{(n-2)/n}$. They also appear in the problem of heat conduction in a nonhomogeneous and nonisotropic medium; in this case $v(x)$ represents the product of the density times the specific heat at the point x and a_{ij} is the thermal conductivity. For results about this equation when $v = 1$ and $w_1 = cw_2$, see for instance [4, 7]. Also, when $v = w_1 = w_2$, see [5, 13].

We present the interpolation inequalities separately because they are of independent interest and because the method we use to prove them is useful in other situations. We give one such result, Landau's inequality, in Theorem 5 below.

We say that a nonnegative and locally integrable function w is a doubling weight if there exists a constant $c > 0$ such that $w(2B) \le cw(B)$ for every ball B in $\mathbf{R}^n$, where $2B$ means the ball with the same center as B and twice the radius. Given $1 < p < \infty$, we say $w \in A_p$ if for all balls B in $\mathbf{R}^n$

$$\left(\frac{1}{|B|}\int_B w\,dx\right)\left(\frac{1}{|B|}\int_B w^{-1/(p-1)}\,dx\right)^{p-1} \le c.$$

We say $w \in A_1$ if $\frac{1}{|B|}\int_B w\,dx \le c\,\operatorname{ess\,inf}_B w$. Also, if v is a weight, $w \in A_p(v)$ means an analogous inequality holds with dx and $|B|$ replaced by $v(x)\,dx$ and $v(B)$, respectively. We also use the notation $A_\infty(v) = \bigcup_{p\ge1} A_p(v)$.

Given $q > 2$, we say that the Poincaré inequality for w_1, w_2 holds for B if there exists a constant $c(B)$ such that for every $u \in \mathrm{Lip}(\overline{B})$ we have

(P)

$$\left(\frac{1}{w_2(B)}\int_B |u - \mathrm{av}_{\mu,B}\,u|^q w_2\,dx\right)^{1/q} \le c(B)\,|B|^{1/n}\left(\frac{1}{w_1(B)}\int_B |\nabla u|^2\,w_1\,dx\right)^{1/2},$$

$$\mathrm{av}_{\mu,B}\,u = \frac{1}{\mu(B)}\int_B u\,d\mu.$$

If we assume w_1^{-1}, w_2 doubling and, for given B,

$$\left(\frac{|\widetilde{B}|}{|B|}\right)^{1/n-1}\left(\frac{w_2(\widetilde{B})}{w_2(B)}\right)^{1/q}\left(\frac{\sigma(\widetilde{B})}{\sigma(B)}\right)^{1/2} \le c \tag{1.2}$$

for every ball $\tilde{B} \subset 2B$ and $\sigma = w_1^{-1}$, then by [14] (P) holds for B with $c(B) = c|B|^{-1}w_1(B)^{1/2}\sigma(B)^{1/2}$, and $\mu = 1$ or $\mu = w_2$. If $w_1 \in A_2$ then (1.2) is equivalent to

$$\left(\frac{|\tilde{B}|}{|B|}\right)^{1/n}\left(\frac{w_2(\tilde{B})}{w_2(B)}\right)^{1/q} \le c\left(\frac{w_1(\tilde{B})}{w_1(B)}\right)^{1/2}. \tag{1.3}$$

Therefore, if $w_1 \in A_2$, w_2 is doubling, and (1.3) holds, then (P) holds with $c(B)$ independent of B and $\mu = 1$ or $\mu = w_2$. We also know that if (P) holds and w_1, w_2, and μ are doubling, then (1.3) holds. In fact, the proof given in [3, p. 1194] for $\mu = 1$ or $\mu = w_2$ works for any doubling μ.

We now state our main results.

Theorem 1. *Let w_1, w_2, and μ be doubling weights. Suppose* (P) *holds for w_1, w_2, and some $q > 2$ for all $B \subset 3B_0$ with center in B_0 and with $c(B)$ independent of B. Suppose also that there exists $c > 0$ such that*

$$\frac{1}{\mu(B)}\int_B |u - \mathrm{av}_{\mu,B}\, u|\mu\, dx \le c|B|^{1/n}\left(\frac{1}{w_1(B)}\int_B |\nabla u|^2 w_1\, dx\right)^{1/2} \tag{1.4}$$

for every $B \subset B_0$ and every $u \in \mathrm{Lip}(\overline{B})$. Then there exists $h > 1$ and a constant c such that

$$\begin{aligned}\frac{1}{w_2(B)}&\int_B |u|^{2h} w_2\, dx \\ &\le c(\mathrm{av}_{\mu,B}|u|)^{2(h-1)}\left(\frac{|B|^{2/n}}{w_1(B)}\int_B |\nabla u|^2 w_1\, dx + (\mathrm{av}_{\mu,B}|u|)^2\right)\end{aligned} \tag{1.5}$$

for every $B \subset B_0$ and every $u \in \mathrm{Lip}(\overline{B})$.

We remark that condition (1.4) with $\mu = w_2$ follows by Hölder's inequality from (P). This is because if (P) holds with any μ then it also holds with $\mu = w_2$. Condition (1.4) also holds for $\mu = 1$ if w_1 satisfies

$$|B|^{-2}\int_B w_1\, dx \int_B w_1^{-1}\, dx \le c$$

for all $B \subset B_0$; see Remark (2.10) in §2 for a proof. See also Remark (2.9) for some comments about the value of h in Theorem 1.

Corollary. *Let w_1 and w_2 be doubling weights. Then there exist $h > 1$ and a constant c such that*

$$\begin{aligned}\frac{1}{w_2(B)}&\int_B |u|^{2h} w_2\, dx \\ &\le c\left(\frac{1}{v(B)}\int_B u^2 v\, dx\right)^{h-1}\left(\frac{|B|^{2/n}}{w_1(B)}\int_B |\nabla u|^2 w_1\, dx + \frac{1}{v(B)}\int_B u^2 v\, dx\right)\end{aligned}$$

for every ball $B \subset B_0$ and every $u \in \mathrm{Lip}(\overline{B})$ under either of the following conditions:

(i) *$w_1, v \in A_2$ and* (P) *holds for w_1, w_2, $\mu = 1$ and some $q > 2$ for all $B \subset 3B_0$ with center in B_0, and with $c(B)$ independent of B.*

(ii) $v/w_2 \in A_2(w_2)$ *and* (P) *holds for* w_1, w_2, *any* μ, *and some* $q > 2$ *for all* $B \subset 3B_0$ *with center in* B_0, *and with* $c(B)$ *independent of* B.

Theorem 2. *Under the same assumptions as in Theorem* 1 *but with* (1.4) *replaced by*

$$\frac{1}{\mu(B)}\int_B |u|\,d\mu \le c|B|^{1/n}\left(\frac{1}{w_1(B)}\int_B |\nabla u|^2 w_1\,dx\right)^{1/2} \tag{1.6}$$

for every $u \in \mathrm{Lip}_0(B)$ *and all* $B \subset 3B_0$, *there exist* $h > 1$ *and a constant* c *such that*

$$\frac{1}{w_2(B)}\int_B |u|^{2h} w_2\,dx \le c(\mathrm{av}_{\mu,B}|u|)^{2(h-1)}\left(\frac{|B|^{2/n}}{w_1(B)}\int_B |\nabla u|^2 w_1\,dx\right) \tag{1.7}$$

for every $B \subset B_0$ *and every* $u \in \mathrm{Lip}_0(B)$.

Theorem 3. *Let* w_1, w_2, *and* μ *be doubling weights and suppose* (P) *holds for* B *with* w_1, w_2, *and some* $q > 2$. *If* $w_2/v \in A_\infty(v)$ *then there exist* $h > 1$ *and a constant* $c > 0$ *such that*

$$\frac{1}{w_2(B)}\int_B |u|^{2h} w_2\,dx$$
$$\le c\left(\frac{1}{v(B)}\int_B u^2 v\,dx\right)^{h-1}\left(\frac{|B|^{2/n}}{w_1(B)}\int_B |\nabla u|^2 w_1\,dx + (\mathrm{av}_{\mu,B}|u|)^2\right)$$

for all $u \in \mathrm{Lip}(\overline{B})$. *Also if* (P) *is replaced by*

$$\left(\frac{1}{w_1(B)}\int_B |u|^q w_2\,dx\right)^{1/q} \le c|B|^{1/n}\left(\frac{1}{w_1(B)}\int_B |\nabla u|^2 w_1\,dx\right)^{1/2} \tag{S}$$

for all $u \in \mathrm{Lip}_0(B)$, *then the inequality above holds with* $(\mathrm{av}_{\mu,B}|u|)^2$ *omitted for all* $u \in \mathrm{Lip}_0(B)$.

See Remark (3.1) for the relation between Theorem 3 and the second part of the corollary to Theorem 1.

The local results proved above lead to global results for functions with compact support in $\mathbf{R}^n$ by letting $B \uparrow \mathbf{R}^n$. For example we have the following theorem.

Theorem 4. *Suppose that* w_1, w_2, *and* v *are doubling weights and that inequality* (1.7) *holds with* $\mu = v$ *for every ball* B *and every* $u \in \mathrm{Lip}_0(B)$ *with constant* c *independent of* u *and* B. *Then*

$$\int_{\mathbf{R}^n} |u|^{2h} w_2\,dx \le c\left(\int_{\mathbf{R}^n} u^2 v\,dx\right)^{h-1}\left(\int_{\mathbf{R}^n} |\nabla u|^2 w_1\,dx\right)$$

for every Lipschitz function u *with compact support if and only if*

$$|B|^{2/n} w_2(B) \le c w_1(B) v(B)^{h-1}$$

for all balls B *in* $\mathbf{R}^n$.

The same method used to prove Theorems 1 and 2 gives the following theorem.

Theorem 5. *Suppose* $1 < p \le q < \infty$, $v \in A_p$, w *is doubling, and the following Poincaré inequality holds: There exists a constant* c *such that for all balls* B

$$\left(\frac{1}{w(B)}\int_B |u - \operatorname{av}_B u|^q \, w\, dx\right)^{1/q} \le c\,|B|^{1/n}\left(\frac{1}{v(B)}\int_B |\nabla u|^p \, v\, dx\right)^{1/p},$$

where $\operatorname{av}_B u$ *is the average of* u *with respect to Lebesgue measure. If* $0 < a < 1$ *then*

$$\|\nabla u\|_{L^q_w} \le c\,\|u\|_{L^p_v}^{1-a}\,\|\nabla^2 u\|_{L^p_v}^{a} \tag{1.8}$$

for every $u \in C^2(\mathbf{R}^n)$ *with compact support if and only if*

$$|B|^{(2a-1)/n}\, w(B)^{1/q} \le c v(B)^{1/p} \tag{1.9}$$

for all balls B *in* $\mathbf{R}^n$. *Here* $\nabla^2 u$ *denotes the vector whose components are the second derivatives of* u.

If $w = v$ and $p = q$ then $v \in A_p$ implies the Poincaré inequality required above and (1.9) means $a = 1/2$. Hence, assuming only that $v \in A_p$, we have

$$\|\nabla u\|_{L^p_v} \le c\,\|u\|_{L^p_v}^{1/2}\,\|\nabla^2 u\|_{L^p_v}^{1/2}.$$

For results about the Landau type inequalities proved in Theorem 5, see [2, 6] and the references listed there.

We remark that Theorems 1–4 and the Corollary have analogues for L^p norms instead of L^2 norms. To obtain the statements of these results, simply replace the exponents 2, $2h$, $2(h-1)$, and $2/n$ by p, ph, $p(h-1)$, and p/n, respectively, at each occurrence, and in (P) and (S) replace the exponent 2 on the right by p, keeping $q > p$. For a related comment, see Remark (2.11).

The organization of the paper is as follows: in §2 we prove Theorems 1 and 2 and the Corollary; in §3 we prove Theorems 3 and 4; and in §4 we prove Theorem 5. For the relation between q and h, see Remarks (2.9) and (3.2).

2. Proof of Theorem 1

The proof is done in two steps. The first step is to show that the conclusion of the theorem holds with the integrals on the right-hand side extended over $3B$, i.e., we will show that there exists $h > 1$ and a constant $c > 0$ such that

$$\begin{aligned}\int_B |u|^{2h}\, w_2\, dx \le c w_2(B)\,(\operatorname{av}_{\mu,3B}|u|)^{2(h-1)} \\ \times\left(\frac{|B|^{2/n}}{w_1(B)}\int_{3B}|\nabla u|^2\, w_1\, dx + (\operatorname{av}_{\mu,3B}|u|)^2\right)\end{aligned} \tag{2.1}$$

for all $B \subset B_0$ and $u \in \operatorname{Lip}(3\overline{B})$. Fix a ball B and let $x \in B$ and B_x be a ball centered at x. We set $q = 2h_0$, $h_0 > 1$. We have

$$\int_{B_x} |u|^{2h}\, w_2\, dy \le 4^h\left\{\int_{B_x} |u - \operatorname{av}_{\mu,B_x} u|^{2h} w_2\, dy + |\operatorname{av}_{\mu,B_x} u|^{2h} w_2(B_x)\right\}.$$

If $h \le h_0$, then by using Poincaré's inequality we have

$$\int_{B_x} |u|^{2h} w_2\, dy$$
$$\le c w_2(B_x) \left\{ |B_x|^{2h/n} \left(\frac{1}{w_1(B_x)} \int_{B_x} |\nabla u|^2 w_1\, dy \right)^h + |\operatorname{av}_{\mu, B_x} u|^{2h} \right\}$$
$$= c w_2(B_x)\{\mathrm{I} + \mathrm{II}\}.$$

Now if B_x is large, i.e., if $B \subset B_x \subset 3B$, then we may assume $\mathrm{I} \ge \mathrm{II}$, because if $\mathrm{I} < \mathrm{II}$ then

$$\int_{B_x} |u|^{2h} w_2\, dy \le c w_2(B_x) |\operatorname{av}_{\mu, B_x} u|^{2h} = c w_2(B_x) |\operatorname{av}_{\mu, B_x} u|^{2(h-1)} (\operatorname{av}_{\mu, B_x} u)^2,$$

which by doubling implies (2.1).

On the other hand, if $|B_x| \to 0$, then $\mathrm{I} \to 0$ and $\mathrm{II} \to |u(x)|^{2h}$. Therefore, if $u(x) \ne 0$, then for small B_x we have $\mathrm{I} < \mathrm{II}$. Hence, given $x \in B$ such that $u(x) \ne 0$, there exists a ball B_x contained in $3B$ such that $\mathrm{I} = \mathrm{II}$, i.e.,

$$|B_x|^{2/n} \left(\frac{1}{w_1(B_x)} \int_{B_x} |\nabla u|^2 w_1\, dy \right) = (\operatorname{av}_{\mu, B_x} u)^2. \tag{2.2}$$

Consequently, for this ball B_x,

$$\int_{B_x} |u|^{2h} w_2\, dy \le c w_2(B_x) |B_x|^{2h/n} \left(\frac{1}{w_1(B_x)} \int_{B_x} |\nabla u|^2 w_1\, dy \right)^h$$
$$= c w_2(B_x) |\operatorname{av}_{\mu, B_x} u|^{2(h-1)} |B_x|^{2/n} \left(\frac{1}{w_1(B_x)} \int_{B_x} |\nabla u|^2 w_1\, dy \right).$$

By Besicovitch's covering lemma there is a family $\{B_k\}_{k=1}^{\infty}$ of such balls which covers B and which has bounded overlaps. Therefore, by adding over k and using the fact that $\operatorname{av}_{\mu, B_k} |u| \le \int_{3B} |u|\, d\mu / \mu(B_k)$ since $B_k \subset 3B$, we obtain

(2.3)
$$\int_B |u|^{2h} w_2\, dx$$
$$\le c \sum_k w_2(B_k) (\operatorname{av}_{\mu, B_k} |u|)^{2(h-1)} \left(\frac{|B_k|^{2/n}}{w_1(B_k)} \int_{B_k} |\nabla u|^2 w_1\, dx \right)$$
$$\le c \left(\int_{3B} |u|\, d\mu \right)^{2(h-1)} \sum_k \frac{1}{\mu(B_k)^{2(h-1)}} \frac{w_2(B_k) |B_k|^{2/n}}{w_1(B_k)} \int_{B_k} |\nabla u|^2 w_1\, dx.$$

We will show that under the hypothesis (P) of Theorem 1, there exists $h_1 > 1$ such that

$$\frac{w_2(\widetilde{B})}{w_2(B)} \le c \left(\frac{\mu(\widetilde{B})}{\mu(B)} \right)^{2(h_1-1)} \left\{ \left(\frac{|B|}{|\widetilde{B}|} \right)^{2/n} \frac{w_1(\widetilde{B})}{w_1(B)} + \left(\frac{\mu(\widetilde{B})}{\mu(B)} \right)^2 \right\} \tag{2.4}$$

for all pairs of balls $\widetilde{B}$, B such that $\widetilde{B} \subset 3B$, $B \subset B_0$, and the center of $\widetilde{B}$ lies in B. Note that if (2.4) holds for some value of h_1 it also holds for all smaller values of h_1. Taking (2.4) momentarily for granted, let us complete the proof of step one with h chosen to be less than or equal to $\min\{h_0, h_1\}$. In fact, using (2.4) with $\widetilde{B} = B_k$, (2.3) is then at most

$$\begin{aligned}
&c\left(\int_{3B} |u|\,d\mu\right)^{2(h-1)} \sum_k \frac{|B_k|^{2/n} w_2(B)}{\mu(B_k)^{2(h-1)} w_1(B_k)} \left(\frac{\mu(B_k)}{\mu(B)}\right)^{2(h-1)} \\
&\quad\times \left\{\left(\frac{|B|}{|B_k|}\right)^{2/n} \frac{w_1(B_k)}{w_1(B)} + \left(\frac{\mu(B_k)}{\mu(B)}\right)^2\right\} \int_{B_k} |\nabla u|^2 w_1\,dx \\
&\leq c w_2(B)(\mathrm{av}_{\mu,3B}|u|)^{2(h-1)} \\
&\quad\times \left\{\frac{|B|^{2/n}}{w_1(B)} \sum_k \int_{B_k} |\nabla u|^2 w_1\,dx \right.\\
&\qquad\qquad \left. + \frac{1}{\mu(B)^2} \sum_k \frac{|B|^{2/n}}{w_1(B_k)} \mu(B_k)^2 \int_{B_k} |\nabla u|^2 w_1\,dx\right\} \\
&\leq c w_2(B)(\mathrm{av}_{\mu,3B}|u|)^{2(h-1)} \\
&\quad\times \left\{\frac{|B|^{2/n}}{w_1(B)} \int_{3B} |\nabla u|^2 w_1\,dx + \frac{1}{\mu(B)^2} \sum_k \left(\int_{B_k} u\,d\mu\right)^2\right\},
\end{aligned}$$

where for the first sum we have used the bounded overlaps of the B_k's, and for the second sum we have used (2.2) for each B_k. Since

$$\begin{aligned}
\frac{1}{\mu(B)} \sum_k \left(\int_{B_k} u\,d\mu\right)^2 &\leq \frac{1}{\mu(B)^2} \left(\sum_k \int_{B_k} |u|\,d\mu\right)^2 \\
&\leq \frac{c}{\mu(B)^2} \left(\int_{3B} |u|\,d\mu\right)^2 \leq c(\mathrm{av}_{\mu,3B}|u|)^2,
\end{aligned}$$

we obtain the desired estimate

$$c w_2(B)(\mathrm{av}_{\mu,3B}|u|)^{2(h-1)} \left\{\frac{|B|^{2/n}}{w_1(B)} \int_{3B} |\nabla u|^2 w_1\,dx + (\mathrm{av}_{\mu,3B}|u|)^2\right\}.$$

We now prove (2.4). Since w_1, w_2, and μ are doubling, (P) implies (1.3) with $q = 2h_0$; therefore

$$\begin{aligned}
\frac{w_2(\widetilde{B})}{w_2(B)} &= \left[\frac{w_2(\widetilde{B})}{w_2(B)}\right]^{1/h_0} \left[\frac{w_2(\widetilde{B})}{w_2(B)}\right]^{1-1/h_0} \\
&\leq c\left(\frac{|B|}{|\widetilde{B}|}\right)^{2/n} \frac{w_1(\widetilde{B})}{w_1(B)} \left[\frac{w_2(\widetilde{B})}{w_2(B)}\right]^{1-1/h_0}.
\end{aligned}$$

Thus, we will be done if there exists $h_1 > 1$ such that

$$\left[\frac{w_2(\widetilde{B})}{w_2(B)}\right]^{1-1/h_0} \leq c \left[\frac{\mu(\widetilde{B})}{\mu(B)}\right]^{2(h_1-1)}.$$

But since w_2 and μ are doubling, there exist ν, $d > 0$ such that $w_2 \in RD_\nu$ and $\mu \in D_d$, i.e.,

$$\mu(\widetilde{B}) \geq c\left(\frac{|\widetilde{B}|}{|B|}\right)^d \mu(B), \qquad w_2(\widetilde{B}) \leq c\left(\frac{|\widetilde{B}|}{|P|}\right)^\nu w_2(B),$$

and consequently

$$\left[\frac{w_2(\widetilde{B})}{w_2(B)}\right]^{1-1/h_0} \leq c\left(\frac{|\widetilde{B}|}{|B|}\right)^{\nu(1-1/h_0)} \leq c\left(\frac{\mu(\widetilde{B})}{\mu(B)}\right)^{(\nu/d)(1-1/h_0)}.$$

Now pick h_1 so close to 1 that $\frac{\nu}{d}(1-1/h_0) \geq 2(h_1-1)$. This completes the proof of step one for any $h \leq \min\{h_0, h_1\}$.

Note that (1.4) is not used in the proof of step one.

We now turn to the second step in the proof of Theorem 1, namely, to deduce the inequality (2.1) for B, B, i.e., to show that

$$\int_B |u|^{2h} w_2\, dx$$
$$\leq c w_2(B) (\mathrm{av}_{\mu, B}|u|)^{2(h-1)} \left(\frac{|B|^{2/n}}{w_1(B)} \int_B |\nabla u|^2 w_1\, dx + (\mathrm{av}_{\mu, B}|u|)^2\right)$$

for all $B \subset \frac{3}{2}B_0$ and $u \in \mathrm{Lip}(\overline{B})$. We use an idea of R. V. Kohn (see [9, 10]).

We construct a family of disjoint balls in B in the following way. Set $B = B_r(x_0)$,

$$\varepsilon_j = \left(\frac{10^3}{1+10^3}\right)^{j-1} \frac{r}{1+10^3}, \qquad j = 1, 2, \ldots,$$

and let R_j be the ring

$$R_j = \{x : \varepsilon_1 + \cdots + \varepsilon_j \leq |x - x_0| < \varepsilon_1 + \cdots + \varepsilon_{j+1}\},$$

$j = 1, 2, \ldots$, $R_0 = B_{\varepsilon_1}(x_0)$. Note that $\bigcup_{j=0}^\infty R_j = B$ since $\sum_{j=1}^\infty \varepsilon_j = r$. The width of R_j is ε_{j+1}. In R_j consider the family of all balls with radius $\frac{1}{2}\varepsilon_{j+1}$ contained in R_j and let $\{B^{kj}\}_k$ be a maximal subfamily of disjoint balls. We set $B^{00} = R_0$.

The family $\{B^{kj}\}_{k,j}$ has the following properties:

(1) $R_j \subset \bigcup_k 3B^{kj}$.

(2) $\lambda B^{kj} \subset B_r(x_0)$ for $\lambda < 10^3 + 1$.

(3) $\{10^2 B^{kj}\}_{k,j}$ has bounded overlaps.

Then by (2.1) applied to each ball $3B^{kj}$

$$\begin{aligned}\int_B |u|^{2h} w_2\,dx &\le \sum_{k,j} \int_{3B^{kj}} |u|^{2h} w_2\,dx \\ &\le c\sum_{k,j} w_2(B^{kj})(\mathrm{av}_{\mu,9B^{kj}}|u|)^{2(h-1)} \\ &\quad\times\left(\frac{|B^{kj}|^{2/n}}{w_1(B^{kj})}\int_{9B^{kj}} |\nabla u|^2 w_1\,dx + (\mathrm{av}_{\mu,9B^{kj}}|u|)^2\right) \\ &= c\sum_{k,j} w_2(B^{kj})(\mathrm{av}_{\mu,9B^{kj}}|u|)^{2(h-1)}\left(\frac{|B^{kj}|^{2/n}}{w_1(B^{kj})}\int_{9B^{kj}} |\nabla u|^2 w_1\,dx\right) \\ &\quad + c\sum_{k,j} w_2(B^{kj})(\mathrm{av}_{\mu,9B^{kj}}|u|)^{2h} = \mathrm{I}+\mathrm{II}.\end{aligned}$$

By estimating I as we estimated the sum in (2.3) and using (2) and (3) we obtain

$$\mathrm{I} \le c(\mathrm{av}_{\mu,B}|u|)^{2(h-1)} w_2(B)\frac{|B|^{2/n}}{w_1(B)}\int_B |\nabla u|^2 w_1\,dx.$$

To estimate II, given B^{kj} let $\mathscr{F}(B^{kj}) = \{A_0^{kj}, A_1^{kj}, \ldots, A_j^{kj}\}$ be the chain of balls connecting B^{kj} and B^{00}. By this we mean $A_0^{kj} = B^{00}$, $A_j^{kj} = B^{kj}$, and $2A_r^{kj} \cap 2A_{r+1}^{kj} \neq \varnothing$, $r = 0, 1, 2, \ldots, j-1$. Then

$$\begin{aligned}\mathrm{II} &= c\sum_{k,j} w_2(B^{kj})(\mathrm{av}_{\mu,9B^{kj}}|u - \mathrm{av}_{\mu,9B^{00}} u + \mathrm{av}_{\mu,9B^{00}} u|)^{2h} \\ &\le 4^h c\sum_{k,j} w_2(B^{kj})(\mathrm{av}_{\mu,9B^{kj}}|u - \mathrm{av}_{\mu,9B^{00}} u|)^{2h} \\ &\quad + 4^h c\sum_{k,j} w_2(B^{kj})|\,\mathrm{av}_{\mu,9B^{00}} u|^{2h} \\ &= \mathrm{II}_1 + \mathrm{II}_2.\end{aligned}$$

Clearly, $\mathrm{II}_2 \le cw_2(B)(\mathrm{av}_{\mu,B}|u|)^{2h}$.

To estimate II_1 we write for $x \in 9B^{kj}$

$$|u(x) - \mathrm{av}_{\mu,9B^{00}} u| = \left|u(x) - \mathrm{av}_{\mu,9B^{kj}} u + \sum_{m=1}^{j}(\mathrm{av}_{\mu,9A_m^{kj}} u - \mathrm{av}_{\mu,9A_{m-1}^{kj}} u)\right|.$$

Then

$$\begin{aligned}\mathrm{II}_1 &\le c\sum_{k,j} w_2(B^{kj})(\mathrm{av}_{\mu,9B^{kj}}|u - \mathrm{av}_{\mu,9B^{kj}} u|)^{2h} \\ &\quad + \sum_{k,j} w_2(B^{kj})\left(\sum_{m=1}^{j}|\,\mathrm{av}_{\mu,9A_m^{kj}} u - \mathrm{av}_{\mu,9A_{m-1}^{kj}} u|\right)^{2h} = S_1 + S_2.\end{aligned}$$

S_1 can be estimated using (1.4). In fact, we write

$$S_1 = \sum_{k,j} w_2(B^{kj})(\mathrm{av}_{\mu,9B^{kj}} |u - \mathrm{av}_{\mu,9B^{kj}} u|)^{2(h-1)}(\mathrm{av}_{\mu,9B^{kj}} |u - \mathrm{av}_{\mu,9B^{kj}} u|)^2$$

and since by (2)

$$\mathrm{av}_{\mu,9B^{kj}} |u - \mathrm{av}_{\mu,9B^{kj}} u| \le 2\,\mathrm{av}_{\mu,9B^{kj}} |u| \le \frac{2}{\mu(9B^{kj})}\int_B |u|\,d\mu,$$

then

$$S_1 \le c\left(\int_B |u|\,d\mu\right)^{2(h-1)} \sum_{k,j} w_2(B^{kj})\frac{1}{\mu(B^{kj})^{2(h-1)}}\frac{|B^{kj}|^{2/n}}{w_1(B^{kj})}\int_{9B^{kj}} |\nabla u|^2 w_1\,dx.$$

Again, in the same fashion by which we estimated I, we obtain

$$S_1 \le c(\mathrm{av}_{\mu,B} |u|)^{2(h-1)} w_2(B)\frac{|B|^{2/n}}{w_1(B)}\int_B |\nabla u|^2 w_1\,dx,$$

which is the desired estimate.

To estimate S_2, use Hölder's inequality for sums to obtain

$$\begin{aligned} S_2 &\le c\sum_{k,j} w_2(B^{kj}) j^{2h-1}\sum_{m=1}^{j} |\mathrm{av}_{\mu,9A_m^{kj}} u - \mathrm{av}_{\mu,9A_{m-1}^{kj}} u|^{2h} \\ &= c\sum_{k,j} w_2(B^{kj}) j^{2h-1}\sum_{m=1}^{j} |\mathrm{av}_{\mu,9A_m^{kj}} u - \mathrm{av}_{\mu,9A_{m-1}^{kj}} u|^{2(h-1)} \\ &\quad\times |\mathrm{av}_{\mu,9A_m^{kj}} u - \mathrm{av}_{\mu,9A_{m-1}^{kj}} u|^2. \end{aligned}$$

Using the Poincaré inequality for w_1, w_2 we have

$$\begin{aligned} &\int_{9A_m^{kj}\cap 9A_{m-1}^{kj}} |\mathrm{av}_{\mu,9A_m^{kj}} u - \mathrm{av}_{\mu,9A_{m-1}^{kj}} u|^2 w_2\,dx \\ &\quad\le 2\left(\int_{9A_m^{kj}} |u - \mathrm{av}_{\mu,9A_m^{kj}} u|^2 w_2\,dx + \int_{9A_{m-1}^{kj}} |u - \mathrm{av}_{\mu,9A_{m-1}^{kj}} u|^2 w_2\,dx\right) \\ &\quad\le c|A_m^{kj}|^{2/n}\frac{w_2(A_m^{kj})}{w_1(A_m^{kj})}\int_{9A_m^{kj}} |\nabla u|^2 w_1\,dx \\ &\qquad + c|A_{m-1}^{kj}|^{2/n}\frac{w_2(A_{m-1}^{kj})}{w_1(A_{m-1}^{kj})}\int_{9A_{m-1}^{kj}} |\nabla u|^2 w_1\,dx. \end{aligned}$$

Since $w_2(A_m^{kj}) \approx w_2(A_{m-1}^{kj}) \approx w_2(9A_m^{kj} \cap 9A_{m-1}^{kj})$, we obtain

$$
\begin{aligned}
&|\operatorname{av}_{\mu, 9A_m^{kj}} u - \operatorname{av}_{\mu, 9A_{m-1}^{kj}} u|^2 \\
&\quad \le c \frac{|A_m^{kj}|^{2/n}}{w_1(A_m^{kj})} \int_{9A_m^{kj}} |\nabla u|^2 w_1 \, dx + c \frac{|A_{m-1}^{kj}|^{2/n}}{w_1(A_{m-1}^{kj})} \int_{9A_{m-1}^{kj}} |\nabla u|^2 w_1 \, dx .
\end{aligned}
$$

Thus, as usual,

$$
\begin{aligned}
S_2 \le c &\left(\int_B |u| \, d\mu \right)^{2(h-1)} \\
&\times \sum_{k,j} w_2(B^{kj}) j^{2h-1} \sum_{m=0}^{j} \frac{|A_m^{kj}|^{2/n}}{\mu(A_m^{kj})^{2(h-1)}} \frac{1}{w_1(A_m^{kj})} \int_{9A_m^{kj}} |\nabla u|^2 w_1 \, dx .
\end{aligned}
$$

We write the last sum as

$$
\begin{aligned}
&\sum_j j^{2h-1} \sum_{k:B^{kj}\in R_j} w_2(B^{kj}) \sum_{A\in\mathscr{F}(B^{kj})} \frac{|A|^{2/n}}{\mu(A)^{2(h-1)}} \frac{1}{w_1(A)} \int_{9A} |\nabla u|^2 w_1 \, dx \\
&= \sum_j j^{2h-1} \sum_{k:B^{kj}\in R_j} w_2(B^{kj}) \sum_{m=0}^{j} \sum_{A\in R_m} \chi_{\mathscr{F}(B^{kj})}(A) \\
&\qquad\qquad \times \frac{|A|^{2/n}}{\mu(A)^{2(h-1)} w_1(A)} \int_{9A} |\nabla u|^2 w_1 \, dx \\
&= \sum_j j^{2h-1} \sum_{m=0}^{j} \sum_{A\in R_m} \frac{|A|^{2/n}}{\mu(A)^{2(h-1)}} \frac{1}{w_1(A)} \int_{9A} |\nabla u|^2 w_1 \, dx \\
&\qquad\qquad \times \left\{ \sum_{k:B^{kj}\in R_j} w_2(B^{kj}) \chi_{\mathscr{F}(B^{kj})}(A) \right\} .
\end{aligned}
$$

We claim that there exists $\sigma > 0$ such that

$$
\sum_{\substack{k:B^{kj}\in R_j \\ A\in\mathscr{F}(B^{kj})}} w_2(B^{kj}) \le \frac{c}{(1+\sigma)^{j-m}} w_2(A) \quad \text{if } A \in R_m . \tag{2.5}
$$

We shall momentarily take (2.5) for granted. Observe that if $A \in R_m$ then

$$
\operatorname{radius}(A) = \frac{1}{2} \left(\frac{10^3}{1+10^3} \right)^m \frac{r}{1+10^3} ,
$$

so if we set $\delta = (1+10^3)/10^3$, then $|A| \le cr^n/\delta^{mn} = c|B|/\delta^{mn}$. Now picking $\varepsilon > 0$ such that $1+\sigma = \delta^\varepsilon$, we see that the sum above is bounded by

$$
\sum_j j^{2h-1} \sum_{m=0}^{j} \sum_{A\in R_m} \frac{|A|^{2/n}}{\mu(A)^{2(h-1)}} \frac{1}{\delta^{\varepsilon(j-m)}} \frac{w_2(A)}{w_1(A)} \int_{9A} |\nabla u|^2 w_1 \, dx . \tag{2.6}
$$

We will show below that there exist $h_2 > 1$ and $\alpha > 0$ such that

$$(2.7)\qquad \frac{w_2(\widetilde{B})}{w_2(B)} \le c\left(\frac{\mu(\widetilde{B})}{\mu(B)}\right)^{2(h_2-1)}\left(\frac{|B|}{|\widetilde{B}|}\right)^{2/n-\alpha}\frac{w_1(\widetilde{B})}{w_1(B)}$$

for all balls $\widetilde{B} \subset B$. Note that if this condition holds for some value of h_2, it also holds for all smaller values of h_2. Taking (2.7) for granted and assuming that $h \le \min\{h_0, h_1, h_2\}$, we see that (2.6) is bounded by

$$c\frac{|B|^{2/n} w_2(B)}{\mu(B)^{2(h-1)} w_1(B)} \sum_j j^{2h-1} \sum_{m=0}^{j} \sum_{A\in R_m} \frac{1}{\delta^{\varepsilon(j-m)}} \left(\frac{|A|}{|B|}\right)^{\alpha} \int_{9A} |\nabla u|^2 w_1\,dx$$

$$\le c|B|^{2/n} \frac{w_2(B)}{\mu(B)^{2(h-1)} w_1(B)} \sum_j j^{2h-1}\delta^{-\varepsilon j} \sum_{m=0}^{j} \delta^{\varepsilon m-\alpha m n} \int_B |\nabla u|^2 w_1\,dx.$$

We have

$$\sum_{j=0}^{\infty} j^{2h-1}\delta^{-\varepsilon j} \sum_{m=0}^{j} \delta^{-\alpha m n+\varepsilon m}$$

$$= \sum_{m=0}^{\infty} \delta^{-\alpha m n+\varepsilon m} \sum_{j=m}^{\infty} j^{2h-1}\delta^{-\varepsilon j} \le c\sum_{m=0}^{\infty} \delta^{-\alpha m n} m^{2h-1},$$

which converges since $\alpha > 0$ and $\delta > 1$.

The same sort of argument that we used to prove (2.4) shows that if $w_2 \in RD_\nu$ and $\mu \in D_d$ then (2.7) holds for any $\alpha > 0$ and $h_2 > 1$ which satisfy

$$\nu(1 - 1/h_0) \ge 2d(h_2 - 1) + \alpha.$$

This implies the result.

Proof of (2.5). Fix $A \in R_m$, and for $m \le s \le j$ define

$$Z_s = \bigcup_{\substack{k\,:\,B^{ks}\in R_s \\ A\in\mathcal{F}(B^{ks})}} B^{ks}.$$

Note $Z_m = A$. We will show that there exists $\sigma > 0$ depending only on the doubling constant of w_2 such that

$$(2.8)\qquad w_2\left(\bigcup_{l=s}^{j} Z_l\right) \ge (1+\sigma) w_2\left(\bigcup_{l=s+1}^{j} Z_l\right).$$

In fact, since the width of Z_s is ε_{s+1}, if we let $\lambda = 10^3/(10^3+1)$, we see that the width of $\bigcup_{l=s+1}^{j} Z_l$ is

$$\lambda\varepsilon_{s+1} + \cdots + \lambda^{j-s}\varepsilon_{s+1} \le \frac{1}{1-\lambda}\varepsilon_{s+1} = (1+10^3)\varepsilon_{s+1}.$$

Therefore, letting

$$2(1+10^3)Z_s = \bigcup_{\substack{k : B^{ks} \in R_s \\ A \in \mathcal{F}(B^{ks})}} 2(1+10^3)B^{ks},$$

we have $\bigcup_{l=s+1}^{j} Z_l \subset 2(1+10^3)Z_s$. Then, by doubling,

$$w_2\left(\bigcup_{l=s+1}^{j} Z_l\right) \le w_2(2(1+10^3)Z_s) \le c w_2(Z_s),$$

and (2.8) follows by taking $\sigma = 1/c$. Iteration of (2.8) gives

$$w_2(Z_j) \le \frac{1}{(1+\sigma)^{j-m}} w_2\left(\bigcup_{l=m}^{j} Z_l\right),$$

and since $2(1+10^3)Z_m \supset \bigcup_{l=m+1}^{j} Z_l$ and $Z_m = A$, (2.5) follows by doubling.

Remarks. (2.9) We now make some comments about the values of h in steps one and two of Theorem 1.

Note that if the weights are doubling and (2.1) holds for a given h, then condition (2.4) holds with $h_1 = h$. This follows as usual by choosing an appropriate function u. Conversely, as the proof shows, (2.1) is valid for any $h \le \min\{h_0, h_1\}$ with h_1 as in (2.4). Moreover, the conclusion of Theorem 1 holds for $h \le \min\{h_0, h_1, h_2\}$ with h_2 as in (2.7). In particular, if $w_2 \in RD_\nu$ and $\mu \in D_d$, then (2.1) holds for $h \le \min\{h_0, 1 + \frac{\nu}{2d}(1 - 1/h_0)\}$. In case $w_1 = w_2$ and $\mu \in D_d$, (2.4) holds with $h_1 = 2 + 1/nd$ and (2.7) holds with $h_2 < 1 + 1/nd$.

(2.10) To see that condition (1.4) holds for $\mu = 1$ and $w_1 \in A_2$, note that

$$|u(x) - \mathrm{av}_B\, u| \le c \int_B \frac{|\nabla u(y)|}{|x-y|^{n-1}}\, dy$$

for $x \in B$. Integrating over B we get

$$\begin{aligned}\int_B |u(x) - \mathrm{av}_B\, u|\, dx &\le c \int_B |\nabla u(y)| \int_B \frac{1}{|x-y|^{n-1}}\, dx\, dy \\ &\le c\,|B|^{1/n} \int_B |\nabla u(y)|\, dy.\end{aligned}$$

Hence, since $|B|^{-2} \int_B w_1\, dx \int_B w_1^{-1}\, dx \le c$, (1.4) follows by Schwarz's inequality.

(2.11) The proofs of the L^p analogues of Theorems 1–4 mentioned in the introduction are the same as in the L^2 case, replacing 2 by p whenever 2 appears in an exponent, including those exponents in conditions (2.4) and (2.7).

Proof of the Corollary. If (i) holds, we first note that the hypothesis of Theorem 1 is satisfied for $\mu = 1$; for (1.4) this follows by Remark (2.10) above. The conclusion of the Corollary then follows from the conclusion of Theorem 1 using the fact that $v \in A_2$ implies

$$\frac{1}{|B|}\int_B |u|\,dx \le c\left(\frac{1}{v(B)}\int_B u^2 v\,dx\right)^{1/2}.$$

If (ii) holds, the fact that (P) is true for any μ implies that (P) is also true for $\mu = w_2$, and thus (1.4) holds for $\mu = w_2$. The conclusion of the Corollary now follows from Theorem 1 with $\mu = w_2$ since $v/w_2 \in A_2(w_2)$ implies

$$\frac{1}{w_2(B)}\int_B |u|\,w_2\,dx \le c\left(\frac{1}{v(B)}\int_B u^2 v\,dx\right)^{1/2}.$$

Proof of Theorem 2. The proof of Theorem 2 is similar to that of Theorem 1. If u has support in B we show in the same fashion that for B_x large, i.e., $B \subset B_x \subset 3B$, we may assume $\mathrm{I} \ge \mathrm{II}$, because if $\mathrm{I} < \mathrm{II}$ then

$$\begin{aligned}\int_{B_x} |u|^{2h} w_2\,dx &\le c w_2(B_x)|\operatorname{av}_{\mu,B_x} u|^{2h}\\ &= c w_2(B_x)|\operatorname{av}_{\mu,B_x} u|^{2(h-1)}(\operatorname{av}_{\mu,B_x} u)^2\\ &\le c w_2(B_x)|\operatorname{av}_{\mu,B_x} u|^{2(h-1)}|B_x|^{2/n}\frac{1}{w_1(B_x)}\int_{B_x}|\nabla u|^2 w_1\,dx\end{aligned}$$

by (1.6) applied to B_x. Here we extend u to be 0 outside B. The rest of the proof of step one is the same as in Theorem 1 except that in estimating (2.3) we replace (2.4) by the weaker condition

$$\frac{w_2(\widetilde{B})}{w_2(B)} \le c\left(\frac{\mu(\widetilde{B})}{\mu(B)}\right)^{2(h_1-1)}\left(\frac{|B|}{|\widetilde{B}|}\right)^{2/n}\frac{w_1(\widetilde{B})}{w_1(B)} \tag{2.12}$$

for all $\widetilde{B} \subset 3B$. Thus we obtain (2.1) with the second factor on the right-hand side omitted. Since $\operatorname{supp} u \subset B$ we do not need step two, and Theorem 2 follows.

In passing we note that condition (2.12) with $h_1 = h$ is necessary for the conclusion of Theorem 2; this again follows by choosing an appropriate function u.

3. Proof of Theorem 3

We write

$$\int_B |u|^{2h} w_2\,dx = \int_B (|u|^{2(h-1)}v^{h-1})(u^2 w_2^{1/h_0})(w_2^{1-1/h_0}v^{1-h})\,dx.$$

Applying triple Hölder's inequality with indices $1/(h-1)$, h_0 and r, $h-1+1/h_0+1/r = 1$, we get

$$\begin{aligned}&\int_B |u|^{2h} w_2\,dx\\ &\quad\le \left(\int_B u^2 v\,dx\right)^{h-1}\left(\int_B u^{2h_0} w_2\,dx\right)^{1/h_0}\left(\int_B w_2^{(1-1/h_0)r} v^{(1-h)r}\,dx\right)^{1/r}.\end{aligned}$$

The second term by Poincaré's inequality is bounded by

$$c\left(\int_B |u - \mathrm{av}_{\mu,B}\, u|^{2h_0} w_2\, dx + w_2(B)(\mathrm{av}_{\mu,B}|u|)^{2h_0}\right)^{1/h_0}$$

$$\leq c\left(|B|^{2h_0/n} w_2(B)\left(\frac{1}{w_1(B)}\int_B |\nabla u|^2 w_1\, dx\right)^{h_0} + w_2(B)(\mathrm{av}_{\mu,B}|u|)^{2h_0}\right)^{1/h_0}$$

$$\leq c w_2(B)^{1/h_0}\left[|B|^{2/n}\left(\frac{1}{w_1(B)}\int_B |\nabla u|^2 w_1\, dx\right) + (\mathrm{av}_{\mu,B}|u|)^2\right].$$

To estimate the third term observe that $(h-1)r + 1 = r(1 - 1/h_0)$; then

$$\int_B w_2^{(1-1/h_0)r} v^{(1-h)r}\, dx = \int_B \left(\frac{w_2}{v}\right)^{(h-1)r+1} v\, dx.$$

If $h \to 1$ then r stays bounded and $(h-1)r + 1 \to 1$. Thus, for h near 1, by the reverse Hölder property of w_2/v,

$$\int_B \left(\frac{w_2}{v}\right)^{(h-1)r+1} v\, dx \leq c v(B)\left(\frac{\int_B (w_2/v) v\, dx}{v(B)}\right)^{(h-1)r+1}$$

$$= c\frac{w_2(B)^{(h-1)r+1}}{v(B)^{(h-1)r}}.$$

Consequently,

$$\int_B |u|^{2h} w_2\, dx \leq c\left(\frac{1}{v(B)}\int_B u^2 v\, dx\right)^{h-1} w_2(B)^{1/h_0} w_2(B)^{h-1+1/r}$$

$$\times\left(|B|^{2/n}\left(\frac{1}{w_1(B)}\int_B |\nabla u|^2 w_1\, dx\right) + (\mathrm{av}_{\mu,B}|u|)^2\right)$$

and the inequality in the theorem follows.

If (P) is replaced by (S), then we estimate the second term above using (S) and we obtain the inequality in the theorem with the term $(\mathrm{av}_{\mu,B}|u|)^2$ on the right-hand side omitted.

Remarks. (3.1) We shall see that the hypothesis $w_2/v \in A_\infty(v)$ of Theorem 3 is weaker than the hypothesis $w_2/v \in A_2(w_2)$ of the Corollary: Note that the hypothesis $w_2/v \in A_2(w_2)$ is equivalent to the hypothesis $v/w_2 \in A_2(w_2)$. However, in order to get a summand $\frac{1}{v(B)}\int_B u^2 v\, dx$ instead of $(\mathrm{av}_{\mu,B}|u|)^2$ in the second factor on the right in Theorem 3 with $\mu = w_2$ we need $w_2/v \in A_2(w_2)$.

To show that $\varphi \in A_2(\varphi v)$ implies $\varphi \in A_\infty(v)$ (φ will be w_2/v), note that if $\varphi \in A_2(\varphi v)$, then

$$\left(\frac{1}{v(B)}\int_B \varphi^2 v\, dx\right)^{1/2} \leq c\frac{1}{v(B)}\int_B \varphi v\, dx;$$

that is, φ satisfies a reverse Hölder condition of order two with respect to $v\, dx$, and therefore $\varphi \in A_\infty(v)$.

(3.2) If δ_0 is the reverse Hölder order of w_2/v with respect to v, then the exponent h in Theorem 3 can be taken to be any value $h \le h_0$ such that

$$(h-1)r+1 = \frac{h_0 - 1}{(2-h)h_0 - 1} \le \delta_0 .$$

Proof of Theorem 4. If $u \in \mathrm{Lip}_0(R^n)$ with $\operatorname{supp} u \subset B_0$, then the sufficiency follows by applying (1.7) in B_0, using Schwarz's inequality on the first factor and then applying the hypothesis

$$|B_0|^{2/n} w_2(B_0) \le c w_1(B_0) v(B_0)^{h-1} .$$

For the necessity take $u = 1$ in B, $u = 0$ outside $2B$, and $|\nabla u| \le c/|B|^{1/n}$.

4. Proof of Theorem 5

We first prove the sufficiency of (1.9) for (1.8). Let B be a ball in $\mathbf{R}^n$ and let $\mathrm{av}_B \nabla u$ be the vector average of ∇u over B with respect to Lebesgue measure. By Poincaré's inequality applied to each component of ∇u,

$$\begin{aligned} &\left(\frac{1}{w(B)} \int_B |\nabla u|^q w\, dx\right)^{1/q} \\ &\quad \le \left(\frac{1}{w(B)} \int_B |\nabla u - \mathrm{av}_B \nabla u|^q w\, dx\right)^{1/q} + |\mathrm{av}_B \nabla u| \\ &\quad \le c|B|^{1/n} \left(\frac{1}{v(B)} \int_B |\nabla^2 u|^p v\, dx\right)^{1/p} + |\mathrm{av}_B \nabla u| . \end{aligned}$$

Now we use the inequality (see [11, Lemma, p. 65])

$$|\mathrm{av}_B \nabla u| \le c\left\{|B|^{-1/n} \frac{1}{|B|} \int_B |u|\, dx + |B|^{1/n} \frac{1}{|B|} \int_B |\nabla^2 u|\, dx\right\}$$

with c independent of B and u. Since $v \in A_p$ implies

$$\frac{1}{|B|} \int_B |\nabla^2 u|\, dx \le c\left(\frac{1}{v(B)} \int_B |\nabla^2 u|^p v\, dx\right)^{1/p}$$

and

$$\frac{1}{|B|} \int_B |u|\, dx \le c\left(\frac{1}{v(B)} \int_B |u|^p v\, dx\right)^{1/p} ,$$

we have

$$\begin{aligned} \int_B |\nabla u|^q w\, dx &\le c w(B)\left\{|B|^{1/n}\left(\frac{1}{v(B)} \int_B |\nabla^2 u|^p v\, dx\right)^{1/p}\right. \\ &\qquad \left. + |B|^{-1/n}\left(\frac{1}{v(B)} \int_B |u|^p v\, dx\right)^{1/p}\right\}^q \\ &= c \frac{w(B)}{v(B)^{q/p}} \{\mathrm{I} + \mathrm{II}\}^q . \end{aligned}$$

Let x be a point where $\nabla u(x) \neq 0$. We assume that B has center at x, and we wish to compare the relative sizes of I and II as $|B| \to 0$ and as $|B| \to \infty$. As $|B| \to \infty$, $\mathrm{I} \to \infty$ since $\nabla^2 u \not\equiv 0$, and $\mathrm{II} \to 0$ since $u \in L^p_v(\mathbf{R}^n)$. Consequently, $\mathrm{I} \geq \mathrm{II}$ for large B. Now consider small B, and first assume $u(x) \neq 0$. Then we have for small B that

$$|B|^{p/n} \frac{1}{v(B)} \int_B |\nabla^2 u|^p v\, dy \leq |B|^{-p/n} \frac{1}{v(B)} \int_B |u|^p v\, dy,$$

and therefore $\mathrm{I} \leq \mathrm{II}$ for B small. In case $u(x) = 0$, since $\nabla u(x) \neq 0$ we have $|u(y)| \approx |x - y|$ for all y which lie in a sector Γ of B. Then $(\operatorname{diam} B = r)$

$$\begin{aligned} \mathrm{II} &\geq r^{-1} \left(\int_\Gamma |x-y|^p v\, dy \right)^{1/p} \geq r^{-1} \left(\int_{\substack{r/2 \leq |x-y| \leq r \\ \Gamma}} |x-y|^p v\, dy \right)^{1/p} \\ &\approx \left(\int_{\substack{r/2 \leq |x-y| \leq r \\ \Gamma}} v\, dy \right)^{1/p} \approx v(B)^{1/p} \end{aligned}$$

since v is doubling. On the other hand,

$$\mathrm{I} \leq r \left(\max_B |\nabla^2 u| \right) v(B)^{1/p},$$

and therefore $\mathrm{I} \leq \mathrm{II}$ for $|B|$ small.

Then, if x is a point at which $|\nabla u(x)| \neq 0$, there exists a ball B_x centered at x such that $\mathrm{I} = \mathrm{II}$, i.e., such that

$$|B_x|^{1/n} \left(\int_{B_x} |\nabla^2 u|^p v\, dy \right)^{1/p} = |B_x|^{-1/n} \left(\int_{B_x} |u|^p v\, dy \right)^{1/p}.$$

Therefore, for such x,

$$\begin{aligned} \int_{B_x} |\nabla u|^q w\, dy &\leq c \frac{w(B_x)}{v(B_x)^{q/p}} |B_x|^{(2a-1)q/n} \\ &\quad \times \left(\int_{B_x} |\nabla^2 u|^p v\, dy \right)^{aq/p} \left(\int_{B_x} |u|^p v\, dy \right)^{(1-a)q/p} \\ &\leq c \left(\int_{B_x} |\nabla^2 u|^p v\, dy \right)^{aq/p} \left(\int_{B_x} |u|^p v\, dy \right)^{(1-a)q/p} \end{aligned}$$

by (1.9). For each $N > 0$, by Besicovitch's covering lemma we can select a subfamily of balls B_k with this property which have bounded overlaps and whose union covers $\{x\colon |x| \le N\,,\ \nabla u(x) \ne 0\}$. Then

$$\begin{aligned}
&\int_{|x|\le N} |\nabla u|^q\, w\, dx \\
&= \int_{|x|\le N\,,\,|\nabla u(x)|\ne 0} |\nabla u|^q\, w\, dx \le \sum_k \int_{B_k} |\nabla u|^q\, w\, dx \\
&\le c \sum_k \left(\int_{B_k} |\nabla^2 u|^p v\, dx\right)^{aq/p} \left(\int_{B_k} |u|^p\, v\, dx\right)^{(1-a)q/p} \\
&\le c \left\{\sum_k \left(\int_{B_k} |\nabla^2 u|^p v\, dx\right)^{a} \left(\int_{B_k} |u|^p\, v\, dx\right)^{1-a}\right\}^{q/p} \qquad \text{(since } q \ge p\text{)} \\
&\le c \left(\sum_k \int_{B_k} |\nabla^2 u|^p v\, dx\right)^{aq/p} \left(\sum_k \int_{B_k} |u|^p\, v\, dx\right)^{(1-a)q/p} \\
&\qquad\qquad\qquad\qquad \text{(by Hölder's inequality)} \\
&\le c \|\nabla^2 u\|_{L^p_v}^{aq}\, \|u\|_{L^p_v}^{(1-a)q}
\end{aligned}$$

since the B_k have bounded overlaps. Since the constant c is independent of N the sufficiency follows by letting $N \to \infty$.

For the necessity, let $B_1 = \frac{1}{2}B$, $B_2 = \frac{4}{3}B_1$, and $B_3 = \frac{5}{3}B_1$. Define u to be a smooth function which is zero outside B, $u = 1$ on B_1, and u linear in $B_3 \backslash B_2$ with $|\nabla u| \approx c/|B|^{1/n}$ there. Then it is easy to check that $|\nabla^2 u| \approx c/|B|^{2/n}$ in $\frac{11}{9}B_1 \setminus \frac{10}{9}B_1$. Condition (1.9) follows from (1.8) by choosing this u in (1.8) and using the fact that the weights are doubling.

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